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The ray method for the deflection of a floating flexible platform in short waves

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Abstract

In this paper, we explain how the 'ray method' can be used to describe the deflection, due to short waves, of a very large floating platform in finite or infinite water depth. The elastic properties of the platform are isotropic, but may be distributed inhomogeneously. In the first section, we give a derivation of the equation for the phase and amplitude functions. Then an integro-differential equation for the determination of the deflection is used to find the initial condition for amplitude along the characteristics. For the homogeneous two-dimensional platform in water of finite depth, an exact solution in the form of a superposition of modes can be obtained. This simplified problem serves as a 'canonical' problem for problems with the same structure locally. In the last section, we give some result for a semi-infinite platform with varying elasticity coefficient, the mass distribution being taken constant.

Keywords: Asymptotic methods; Boundary-integral equations; Free-surface flows; Hydroelasticity; Ray method

1. Introduction

One of the concepts for the design of very large floating airports, is to build a very large mat-like structure, that is kept on station by means of anchor lines or by means of a dynamic positioning system. One of the advantages of this kind of structure is that it can be towed towards its temporary destination. Ohkusu and Nanba (1996) presented an asymptotic theory to describe the deflection of the platform due to relatively short incident waves while it is positioned in shallow water. Due to the fact that the vertical dimension is averaged out it is relatively simple to derive a valid formulation. Hermans (1997) derived a formulation for deep water. Unfortunately, this formulation uses some nonphysical boundary conditions and its applicability is questionable for this reason. Later, Hermans (2000) derived an exact differential–integral formulation for the deflection. In the latter paper, the problem is solved numerically. In a subsequent paper, this differential–integral equation is used to obtain a short-wave solution for the homogeneous semi-infinite platform, positioned in deep water. In this paper, the method is extended to the determination of the deflection of short waves of a platform of general shape and inhomogeneous elastic properties.

2. Mathematical formulation

In this section, we derive the general formulation for the diffraction of waves by a flexible platform of general geometric form.

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The fluid is incompressible, so we introduce the velocity potential $\boldsymbol{V}(\boldsymbol{x}, t) = \nabla\Phi(\boldsymbol{x}, t)$, where $\boldsymbol{V}(\boldsymbol{x}, t)$ is the fluid velocity vector. We assume waves in still water. Hence, $\Phi(\boldsymbol{x}, t)$ is a solution of the Laplace equation

$$\Delta\Phi = 0 \quad \text{in the fluid,} \tag{1}$$

together with the linearized kinematic condition, $\Phi_z = w_t$, and dynamic condition, $p/\rho = -\Phi_t - gw$, at the linearized free water surface $z = 0$, where $w(x, y, t)$ denotes the free surface elevation, and ρ is the density of the water. The linearized free surface condition outside the platform becomes

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \tag{2}$$

at $z = 0$ and $(x, y)\in\mathscr{F}$.

The platform is assumed to be a thin layer at the free surface $z = 0$, which seems to be a good model for a shallow draft platform. The platform is modelled as an elastic plate with zero thickness. To describe the deflection $w(x, y)$, we apply the isotropic thin plate theory, which leads to an equation for w of the form

$$m(x, y)\frac{\partial^2 w}{\partial t^2} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(D(x, y)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right) + p|_{z=0}, \tag{3}$$

where $m(x, y)$ is the mass of unit area of the platform while $D(x, y)$ is its equivalent flexural rigidity. We apply the operator $\partial/\partial t$ to Eq. (3) and use the kinematic and dynamic condition to arrive at the following equation for Φ at $z = 0$ and in the platform area $(x, y)\in\mathscr{P}$:

$$\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{D(x, y)}{\rho g}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) + \frac{m(x, y)}{\rho g}\frac{\partial^2}{\partial t^2} + 1\right\}\frac{\partial\Phi}{\partial z} + \frac{1}{g}\left\{\frac{\partial^2}{\partial t^2}\right\}\Phi = 0. \tag{4}$$

The free edges of the platform are free of shear forces and moment. We assume that the flexural rigidity is constant along the edge and its normal derivative equals zero. Also, we assume that the radius of curvature, in the horizontal plane, of the edge is large. Hence, the edge may be considered to be straight locally. We then approximate the boundary conditions at the edge by

$$\frac{\partial^2 w}{\partial n^2} + v\frac{\partial^2 w}{\partial s^2} = 0 \quad \text{and} \quad \frac{\partial^3 w}{\partial n^3} + (2 - v)\frac{\partial^3 w}{\partial n\partial s^2} = 0, \tag{5}$$

where v is Poisson's ratio, n is in the normal direction, in the horizontal plane, along the edge and s denotes the arc length along the edge. At the bottom of the fluid region $z = -h$ we have

$$\frac{\partial\Phi}{\partial z} = 0. \tag{6}$$

The harmonic wave can be written as $\Phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x})e^{-i\omega t}$. Due to the large-length scales and elastic parameters involved we introduce dimensionless coordinates and parameters in the following way:

$$\boldsymbol{x}' = \frac{\boldsymbol{x}}{L}, \quad h' = \frac{h}{L}, \quad K = \frac{\omega^2 L}{g}, \quad \mu = \frac{m\omega^2}{\rho g}, \quad \mathscr{D} = \frac{DK^4}{L^4\rho g}.$$

The parameters μ and $\mathscr{D}$ are of order one for large values of K. In a practical situation, where L is of the order of 1000 m and a normal sea spectrum, this is the case. Later, we take, for convenience, $L = 300$ m as a characteristic length also for the semi-infinite plate. After dropping the primes we obtain at $z = 0$:

$$\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\mathscr{D}(x, y)}{K^4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) - \mu(x, y) + 1\right\}\frac{\partial\phi}{\partial z} - K\phi = 0. \tag{7}$$

The potential of the undisturbed incident wave is given by

$$\phi^{\text{inc}}(\boldsymbol{x}) = \frac{g\zeta_\infty}{i\omega}\frac{\cosh(k_0(z + h))}{\cosh(k_0 h)}\exp\{ik_0(x\cos\beta + y\sin\beta)\}, \tag{8}$$

where ζ_∞ is the wave height in the original coordinate system, ω the frequency, while the wavenumber obeys the dispersion relation, $k_0\tanh(k_0 h) = K$, for finite water depth. We continue with the deep water case $h = \infty$; hence $K = k_0 = \omega^2 L/g$, and we assume that the potential underneath the plate can be written as a superposition of *ray*-mode solutions as follows:

$$\phi(\boldsymbol{x}, K) = \sum_n \alpha_n(\boldsymbol{x}, K)\mathrm{e}^{\mathrm{i}KS_n(\boldsymbol{x})}, \tag{9}$$

where $S_n(\boldsymbol{x})$ is the phase function and $\alpha_n(\boldsymbol{x}, K)$ the amplitude function of the nth mode. In Eq. (9), each mode is written as a regular series expansion with respect to inverse powers of iK,

$$\alpha_n(\boldsymbol{x}, K) = \sum_{j=0}^{N} \frac{\alpha_{n,j}(\boldsymbol{x})}{(\mathrm{i}K)^j} + o((\mathrm{i}K)^{-N}). \tag{10}$$

We now drop the index n of the mode for a while. Insertion of Eq. (9) into the Laplace equation (1) gives

$$-K^2\alpha\nabla_3 S \cdot \nabla_3 S + \mathrm{i}K(2\nabla_3\alpha \cdot \nabla_3 S + \alpha\Delta_3 S) + \mathcal{O}(1) = 0. \tag{11}$$

Subscript 3 is used to indicate the three-dimensional (3-D) ∇ and Δ operators. If no subscripts are used, the operators are two dimensional (2-D) in the horizontal plane. Next we insert Eq. (10) and compare orders of magnitude in Eq. (11). This leads to a set of equations for S and α_0 to be satisfied in the fluid region:

$$\mathcal{O}(K^2):\ \ \nabla_3 S \cdot \nabla_3 S = 0, \tag{12}$$

$$\mathcal{O}(K^1):\ 2\nabla_3\alpha_0 \cdot \nabla_3 S + \alpha_0\Delta_3 S = 0. \tag{13}$$

We now insert Eq. (9) into the condition at $z = 0$ (7). The first two terms in the expansion become

$$\mathcal{O}(K^1):\ \{\mathcal{D}(x,y)(S_x^2 + S_y^2)^2 - \mu(x,y) + 1\}\mathrm{i}S_z = 1 \tag{14}$$

and $\mathcal{O}(K^0)$:

$$\begin{aligned}\alpha_0\mathcal{D}&\left[\frac{\partial}{\partial z}(S_x^2 + S_y^2)^2 + 2S_z\left\{\frac{\partial}{\partial x}S_x(S_x^2 + S_y^2) + \frac{\partial}{\partial y}S_y(S_x^2 + S_y^2)\right\}\right]\\ &+\ \alpha_{0z}\{\mathcal{D}(x,y)(S_x^2 + S_y^2)^2 - \mu(x,y) + 1\} + (4\mathcal{D}\nabla\alpha_0 \cdot \nabla S + 2\alpha\nabla\mathcal{D} \cdot \nabla S)S_z(S_x^2 + S_y^2) = 0.\end{aligned} \tag{15}$$

If we write $r = \mathrm{i}S_z$ and combine Eq. (12) with Eq. (14), we obtain the dispersion relation at $z = 0$:

$$(\mathcal{D}(x,y)r^4 - \mu(x,y) + 1)r = 1 \tag{16}$$

combined with

$$S_x^2 + S_y^2 = r^2. \tag{17}$$

The last equation has the same form as the well-known *eikonal* equation in geometrical optics; however, in this case the right-hand side is given by an implicit relation (16). In the case of constant elastic coefficients r is a constant and the rays are straight lines as may be expected. We assume that there is a propagating wave solution with a real-valued phase function S. For constant values of μ, the characteristics (*rays*) become

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}\sigma} = \mathcal{F}S_x, \quad \frac{\mathrm{d}y}{\mathrm{d}\sigma} = \mathcal{F}S_y,\\ &\frac{\mathrm{d}S_x}{\mathrm{d}\sigma} = -\mathcal{D}_x r^5, \quad \frac{\mathrm{d}S_y}{\mathrm{d}\sigma} = -\mathcal{D}_y r^5, \quad \frac{\mathrm{d}S}{\mathrm{d}\sigma} = \mathcal{F}r^2\end{aligned} \tag{18}$$

with $\mathcal{F} = (5\mathcal{D}r^4 - \mu + 1)/r$ and σ the parameter along the ray.

To obtain an equation for the amplitude α_0, at $z = 0$, we use the equation in fluid (13) to eliminate the z-derivatives in Eq. (15). The second-order derivative S_{zz} is obtained by means of differentiation with respect to z of Eq. (12). We finally get for variable $\mathcal{D}$ and constant μ:

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}\sigma} = -\alpha_0 M\{S\}, \tag{19}$$

where the operator $M\{S\}$ is defined as:

$$M\{S\}(x,y) = \frac{(\mathrm{d}r/\mathrm{d}\sigma)\{16r^4\mathcal{D} - 1/r\} - \mathcal{F}(S_{xx} + S_{yy})(4\mathcal{D}r^5 + 1) + 4r^4(\partial\mathcal{D}/\partial\sigma)}{8\mathcal{D}r^5 + 2}. \tag{20}$$

In principle, we can solve these equations if initial conditions for the wave modes are available. We have one travelling wave mode and two evanescent modes. The problem, however, is that we cannot derive a set of initial conditions for the amplitudes. One should think of writing the field outside the platform as a superposition of an incident and a reflected wave. So we have four unknown coefficients to determine, while there are only two conditions at the edge of the plate. One may try to impose some matching conditions, such as continuity of velocity and potential.

Such conditions hold in the fluid domain and not only at $z = 0$. The fact that the z-dependance of the potentials described is different underneath the platform and outside the platform makes it impossible to succeed in matching the two fields. In the next section, a differential–integral formulation at $z = 0$ will be derived that obeys the necessary continuity conditions. This analysis can be carried out analytically for the case of a flexible strip in finite water depth. The computed results, for sufficiently large value of the water depth, may serve as initial conditions for the ray solutions described above if the geometry is similar locally.

3. Differential–integral formulation

In this section, the solution for the half-plane and the strip will be derived. To do so, the fluid domain will be split into two regions. We define the region underneath the platform as $\mathscr{D}^-$ and the region towards infinity as $\mathscr{D}^+$, while the interface is denoted by $\partial\mathscr{D}$ (see Fig. 1). The potential function in $\mathscr{D}^+$ is written as a superposition of the incident wave potential and a diffracted wave potential, as follows:

$$\phi(\boldsymbol{x}) = \phi^{\text{inc}}(\boldsymbol{x}) + \phi^+(\boldsymbol{x}), \tag{21}$$

while the total potential in $\mathscr{D}^-$ is denoted by $\phi^-(\boldsymbol{x})$. It will be shown that this choice leads to an interesting way to derive an integral equation, that can be solved numerically. At the dividing surface $\partial\mathscr{D}$ we require continuity of the total potential and its normal derivative.

We introduce the Green's function $\mathscr{G}(\boldsymbol{x}, \xi)$ that fulfils $\Delta\mathscr{G} = 4\pi\delta(\boldsymbol{x} - \xi)$, the boundary conditions at the free surface and the bottom and also the radiation condition. Green's functions for several free surface problems can be found in Wehausen and Laitone (1960).

We apply Green's theorem to the potential ϕ^+ and ϕ^-, respectively. This leads to the following approach. For $\boldsymbol{x} \in \mathscr{D}^+$, we have

$$\begin{aligned} 4\pi\phi^+ &= -\int_{\partial\mathscr{D}\cup\mathscr{F}} \left(\phi^+\frac{\partial\mathscr{G}}{\partial n} - \mathscr{G}\frac{\partial\phi^+}{\partial n}\right) \mathrm{d}S, \\ 0 &= \int_{\partial\mathscr{D}\cup\mathscr{P}} \left(\phi^-\frac{\partial\mathscr{G}}{\partial n} - \mathscr{G}\frac{\partial\phi^-}{\partial n}\right) \mathrm{d}S \end{aligned} \tag{22}$$

and in the region $\boldsymbol{x} \in \mathscr{D}^-$ we have

$$\begin{aligned} 0 &= -\int_{\partial\mathscr{D}\cup\mathscr{F}} \left(\phi^+\frac{\partial\mathscr{G}}{\partial n} - \mathscr{G}\frac{\partial\phi^+}{\partial n}\right) \mathrm{d}S, \\ 4\pi\phi^- &= \int_{\partial\mathscr{D}\cup\mathscr{P}} \left(\phi^-\frac{\partial\mathscr{G}}{\partial n} - \mathscr{G}\frac{\partial\phi^-}{\partial n}\right) \mathrm{d}S. \end{aligned} \tag{23}$$

The integrals over $\mathscr{F}$ become zero, due to the zero current free surface condition for $\mathscr{G}$ and ϕ^+. We add up the two expressions in Eq. (23) and use the free surface condition for the Green's function and the potential ϕ^+,

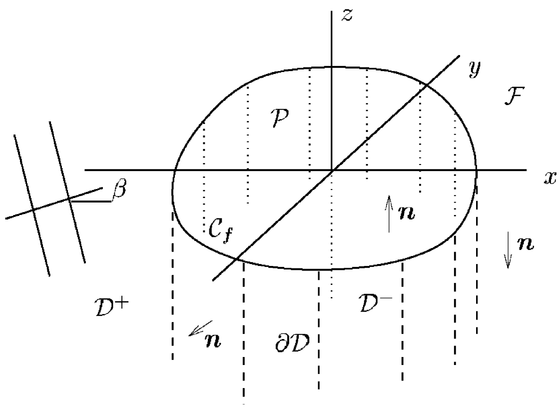

Fig. 1. Definition of 3-D geometry.

leading to

$$4\pi\phi^- = \int_{\partial\mathscr{D}} \left([\phi]\frac{\partial\mathscr{G}}{\partial n} - \mathscr{G}\left[\frac{\partial\phi}{\partial n}\right] \right) \mathrm{d}S + \int_{\mathscr{P}} (K\phi^- - \phi^-_\zeta)\mathscr{G}\,\mathrm{d}S \quad \text{for } \boldsymbol{x}\in\mathscr{D}^-, \tag{24}$$

where we used the notation $[\cdots]$ for the jump of the function concerned. Furthermore, we use the jump condition between the potentials ϕ^+ and ϕ^- and their derivatives. For the total potential the jumps are zero, so we obtain

$$4\pi\phi^- = \int_{\partial\mathscr{D}} \left(\phi^{\text{inc}}\frac{\partial\mathscr{G}}{\partial n} - \mathscr{G}\frac{\partial\phi^{\text{inc}}}{\partial n} \right) \mathrm{d}S - \int_{\mathscr{P}} \left(\mu(\xi,\eta)\phi^-_\zeta - \left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\mathscr{D}(\xi,\eta)}{K^4}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\right)\phi^-_\zeta \right)\mathscr{G}\,\mathrm{d}S, \tag{25}$$

where we have used relation (4) for ϕ^- at the platform. Relation (25) is suitable for further manipulation to end up with a differential–integral equation, that can be solved numerically. The Green's function itself has a weak singularity, so we may take the limit $z\to 0$ and use Eq. (4) to express ϕ^- in terms of an operator acting on ϕ^-_z. Furthermore, we notice that the first integral on the right-hand side of Eq. (25) can be simplified significantly. This term is independent of the parameters of the platform, hence, it is the same if there is no platform present. Therefore, it equals $4\pi\phi^{\text{inc}}$. This can also be verified by manipulating the integrals. At $z=0$, we arrive at the following equation for the deflection of the plate,

$$4\pi\left\{1-\mu(x,y)+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\mathscr{D}(x,y)}{K^4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right\}w(x,y)+K\int_{\mathscr{P}}\mathscr{G}(x,y;\xi,\eta)\left\{\mu(\xi,\eta)-\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\right.$$
$$\left.\times\;\frac{\mathscr{D}(\xi,\eta)}{K^4}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\right\}w(\xi,\eta)\,\mathrm{d}S = 4\pi\zeta^\infty\exp\{\mathrm{i}k_0(x\cos\beta+y\sin\beta)\}. \tag{26}$$

In Hermans (2000), Eq. (26), with constant coefficients, together with its boundary conditions (5) has been solved numerically.

To find the initial conditions for the phase function and the amplitude in the *ray* solution, we restrict ourselves to a homogeneous half-plane at finite water depth. Hence, the boundary conditions become

$$\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}=\frac{\partial^3 w}{\partial x^3}+(2-\nu)\frac{\partial^3 w}{\partial x\partial y^2}=0 \quad \text{at } x=0. \tag{27}$$

For large values of x, the solution is finite and consists of outgoing waves; at $z=\zeta=0$, a 2-D Green's function for finite water depth, obeying the radiation condition. It has the form.

$$\mathscr{G}(x,\xi) = -\int_{\mathscr{L}'}\frac{\cosh kh}{k\sinh kh - K\cosh kh}\mathrm{e}^{\mathrm{i}k(x-\xi)}\,\mathrm{d}k. \tag{28}$$

One obtains this form if one writes the Green's function in the fluid as the difference of a logarithmic singularity and its mirror with respect to the free surface instead of the bottom, as is commonly done, see Wehausen and Laitone (1960). The remaining integral contribution reduces to Eq. (28) at the free surface. Expression (28) will be used in the case of $\beta=0$ and the following 3-D version is used in the general case:

$$\mathscr{G}(x,y;\xi,\eta) = -2\int_0^\infty\frac{k\cosh kh}{k\sinh kh - K\cosh kh}J_0(kR)\,\mathrm{d}k. \tag{29}$$

The contour of integration $\mathscr{L}'$ is given in Fig. 2, while in Eq. (29) the contour is the positive part of $\mathscr{L}'$. It is chosen such that the radiation condition is fulfilled. R is the horizontal distance, so $R^2=(x=\xi)^2+(y-\eta)^2$. For the half-plane problem, the solution will be sought as a superposition of exponential functions of the form

$$w(x,y)=\frac{\mathrm{i}}{\omega}\phi^-_z|_{z=0}=\sum_n a_n\exp\{\mathrm{i}\kappa_n x+\mathrm{i}k_0 y\sin\beta\} \quad \text{for } 0\leqslant\beta\leqslant\beta_{cr}<\frac{\pi}{2}, \tag{30}$$

where for the platform with homogeneous physical properties it is expected that the constant 'amplitudes' a_n and 'wavenumbers' κ_n can be determined for angles of incidence smaller than a, so far unknown, critical angle β_{cr}. Due to the fact that we consider a half-plane, the real part of κ_n has to be negative or if the real part equals zero it must obey the outgoing condition. The inhomogeneous term in the equation behaves like $\exp(\mathrm{i}k_0x)$; this does not indicate that the solution behaves accordingly, as we will see.

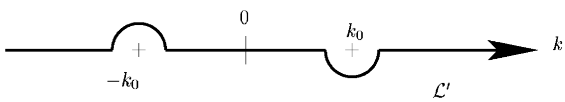

Fig. 2. Contour of integration.

We make use of a Sonine–Gegenbauer expression for Bessel functions, to carry out the integration with respect to η:

$$\int_0^\infty \cos(bt) J_0\{k\sqrt{a^2+t^2}\}\,\mathrm{d}t = \begin{cases} 0 & \text{if } k<b, \\ \dfrac{\cos(a\sqrt{k^2-b^2})}{\sqrt{k^2-b^2}} & \text{if } k>b \end{cases} \tag{31}$$

and we work out the integration with respect to ξ to obtain

$$\begin{aligned} &\sum_n a_n\left(\frac{\mathscr{D}}{K^4}\kappa^{(n)^4}-(\mu-1)\right)\mathrm{e}^{\mathrm{i}\kappa_n x} \\ &\quad = \mathrm{i}\sum_n a_n \frac{K}{4\pi}\left(\frac{\mathscr{D}}{K^4}\kappa^{(n)^4}-\mu\right)\int_{\mathscr{L}'}\frac{\cosh kh}{k\sinh kh - K\cosh kh}\left(\frac{\mathrm{e}^{\mathrm{i}x\sqrt{k^2-k_0^2\sin^2\beta}}}{\sqrt{k^2-k_0^2\sin^2\beta}-\kappa_n}-\frac{\mathrm{e}^{-\mathrm{i}x\sqrt{k^2-k_0^2\sin^2\beta}}}{\sqrt{k^2-k_0^2\sin^2\beta}+\kappa_n}\right) \\ &\quad\quad \times \frac{k\,\mathrm{d}k}{\sqrt{k^2-k_0^2\sin^2\beta}}+\zeta_\infty \mathrm{e}^{\mathrm{i}k_0 x\cos\beta}. \end{aligned} \tag{32}$$

Here, we define $\kappa^{(n)}$ as follows:

$$\kappa^{(n)^2} = \kappa_n^2 + k_0^2\sin^2\beta.$$

The coefficient b in expression (31) corresponds to $k_0\sin\beta$. The contribution of the integral along the branchcut must be zero. This results in the choice of two different contours. For the first part of the integrand, the contour is chosen above the branchcut, while for the second part with the minus sign in the exponential, the contour is chosen underneath the branchcut. Adding up the two integrals results in a zero contribution in accordance with Eq. (31). The first part of the integral can be closed in the upper half-plane. The final result is an integral which can be written as a sum of residues. Until now the values of κ_n are still unknown. We assume that the poles at $\sqrt{k^2-k_0^2\sin^2\beta}=\kappa_n$ are in the upper half-plane. Application of the residue lemma at these points leads to the dispersion relation for $\kappa^{(n)}$:

$$\left(\frac{\mathscr{D}}{K^4}\kappa^4-\mu+1\right)\kappa\tanh\kappa h = K. \tag{33}$$

This is for the deep water case the same as Eq. (16) with $\kappa = Kr$. This dispersion relation has solutions in the complex plane, at the real axis $\pm\kappa^{(0)}$, at the imaginary axis $\pm\kappa^{(n)}$, $n=3,4,5,\ldots$, and four in the complex plane $\pm\kappa^{(1,2)} = \pm(\kappa_{re}\pm \mathrm{i}\kappa_{im})$. In our ray expansions, only those values that obey the radiation condition play a role, hence the contour of integration passes underneath the pole on the positive real axis and above the one on the negative real axis. One must keep in mind that the position of the poles is similar to that for the dispersion relation for the water surface, except for the two extra complex poles in the upper complex plane.

We now have to consider the zeros of the dispersion relation for the water surface $k\tanh kh = K$. These lead to the relations for the determination of the amplitudes a_n.

We truncate the series in the ray expansion at $N+3$ terms, this means that we have to take into account $N+1$ zeros of the water dispersion relation, one on the real and N imaginary. If one closes the contour in the complex k-plane, the contribution of these poles leads to

$$\sum_{n=0}^{N+2}\frac{K[(\mathscr{D}\kappa^{(n)^4}/K^4)-\mu]k_0 a_n}{(K(1-Kh)+k_0^2 h)\cos\beta(\kappa_n-k_0\cos\beta)}+\zeta_\infty = 0 \tag{34}$$

and for $i=1,\ldots,N$:

$$\sum_{n=0}^{N+2}\frac{K[(\mathscr{D}\kappa^{(n)^4}/K^4)-\mu]k_i^2 a_n}{(K(1-Kh)+k_i^2 h)\sqrt{k_i^2-k_0^2\sin^2\beta}\,(\kappa_n-\sqrt{k_i^2-k_0^2\sin^2\beta})} = 0. \tag{35}$$

Together with the two boundary conditions at the edge of platform Eq. (27), we have $N+3$ equations for the $N+3$ unknown values of a_n. The solutions of this set of equations serve as initial conditions for the 'ray' expansion for the inhomogeneous case.

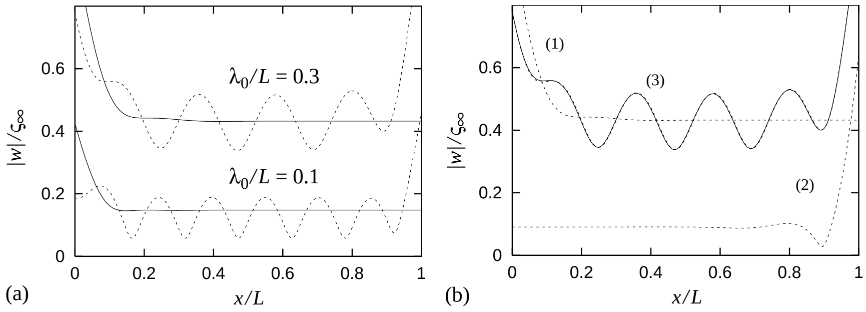

Fig. 3. Example with parameters $D/\rho g = 1.23 \times 10^{-5} \times L^4$, $h/L = 1/3$ and $\beta = 0$. (a) Asymptotic results for finite and semi-infinite (dotted) platform. (b) Multiple reflection $\lambda_0/L = 0.3$, (1) semi-inf., (2) reflected at $x = L$, (3) sum of semi-inf. and refl. at $x = L$ and 0.

The approach described in this section can be extended to the case of a strip of width L. In dimensionless parameters, the solution is written as

$$w(x, y) = \sum_{n=0}^{N} (a_n \exp\{i\kappa_n x\} + b_n \exp\{-i\kappa_n(x-1)\})\exp\{ik_0 y \sin\beta\}. \tag{36}$$

In Fig. 3(a), we give some results for a semi-infinite platform and a strip with width L at two values of the wavelength.

Due to the rather simple structure of the strip problem the results shown are exact solutions. In the context of the 'ray' method, one is interested in the propagation along a ray, while in the exact results multiple reflections are taken into account. In Fig. 3(b), results for the propagation of the rays are shown with zero reflections, the contribution of one reflection at $x = L$ separately, the combined result after one reflection at $x = L$ and one at $x = 0$ and the result after multiple reflection. In Fig. 4, we compare our results with the ones of Takagi. He compared two methods, different from ours, with each other. Fig. 4 shows the results of the three methods. No differences can be distinguished, so we conclude that all three methods lead to the same result.

Some results of reflection and transmission coefficients are shown in Fig. 5. In all these examples, we also tested the influence of the number N. It turns out that $N = 10$ is sufficient to acquire a high degree of accuracy. For most cases, $N = 3$ is sufficient. This means that in the 'water'-dispersion relation only k_0 is taken into account and in the 'plate'-dispersion relation the progressing wave mode and two evanescent modes are taken into account.

4. Boundary conditions for the 'ray' method

We now study the inhomogeneous case. The 2-D deep water case will be studied here. The equation now becomes

$$2\pi\left\{1 - \mu(x) + \frac{d^2}{dx^2}\frac{\mathscr{D}(x)}{K^4}\frac{d^2}{dx^2}\right\}w(x) + K\int_{\mathscr{P}} \mathscr{G}(x;\xi)\left\{\mu(\xi) - \frac{d^2}{d\xi^2}\frac{\mathscr{D}(\xi)}{K^4}\frac{d^2}{d\xi^2}\right\}w(\xi)\, dS = 2\pi\zeta^\infty \exp\{ik_0 x\}, \tag{37}$$

where the Green's function is written as

$$\mathscr{G}(\boldsymbol{x}, \boldsymbol{\xi}) - \ln r = -\ln r_1 - 2\int_{\mathscr{L}'} \frac{1}{k - k_0} e^{k(z+\zeta)} \cos k(x - \xi)\, dk, \tag{38}$$

where the contour $\mathscr{L}'$ is in the complex k-plane from $k = 0$ to ∞ that passes, due to the radiation condition, underneath the pole of the integrand $k = k_0 = K = \omega^2/g$. We have seen that the solution of the dispersion relation (16) gives rise to three modes. To find initial conditions for the 'transport' equation (19) we insert, in Eq. (37), the ray *Ansatz* for the deflection following from Eq. (9),

$$w(x, K) = \sum_n a_n(x) e^{iKS_n(x)}$$

with $a_n = -(KS_z/\omega)\alpha_n$. In the uniform case, we have seen that the amplitudes are determined by means of the contribution of the pole in the integrand, $k = k_0$. If we use the dispersion relation to determine the phase function, the final result is an integral equation for the amplitude. This equation may be solved numerically. However, if we follow

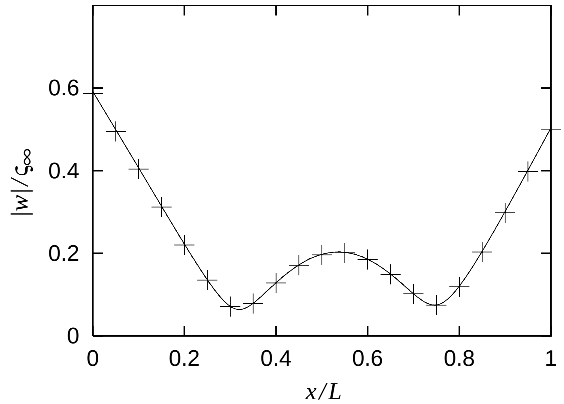

Fig. 4. Comparison with the results of Takagi for $\lambda_0/L = 0.278$ and $D/\rho g = 1.74 \times 10^{-3} \times L^4$, $h/L \approx \frac{1}{3}$ and $\beta = 0$.

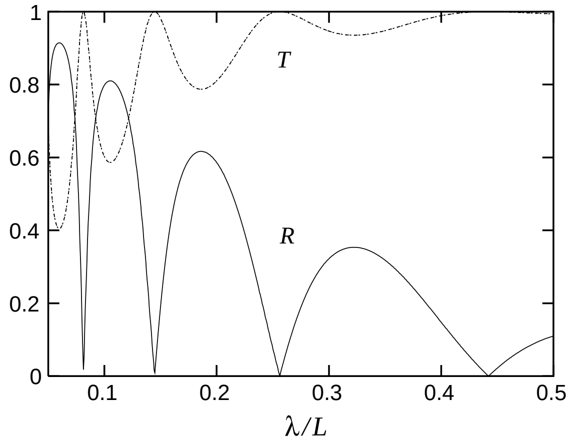

Fig. 5. Reflection and transmission coefficients $D/\rho g = 1.23 \times 10^{-5} \times L^4$, $h/L = \frac{1}{3}$ and $\beta = 0$.

this approach it is much easier to solve the original equation directly. If we carry out partial integration with respect to ξ the main contribution at $x = 0$ leads to a relation for the initial conditions, see Hermans (2002) of the amplitude functions:

$$\sum_n \frac{K(\mathscr{D}(0)S_\xi^4 - \mu(0))}{k_0 - KS_\xi} a_n = \zeta_\infty. \tag{39}$$

It also can be shown that if the integration with respect to ξ is carried out first the contribution of the poles $k = KS_\xi$ give rise to the correct dispersion relation. In the case of infinite water depth, the contribution of the integration along the imaginary axis in the complex k plane has to be taken into account iteratively, as is shown by Hermans (2001). The influence of this integral is small. In the case of finite depth, more wave modes have to be taken into account.

In the one-dimensional case with the characteristic equations given in Eq. (18), the evanescent modes can be described by the same equations if one takes x as independent variable instead of σ.

5. Results

In Fig. 6, we show the influence of variation of the elastic parameter along a semi-infinite platform. The elastic coefficient $d = D/(\rho g)$ is a constant $d^{(1)}$ over the first 15 m and $d^{(2)}$ starting at 25 m from the origin, between these

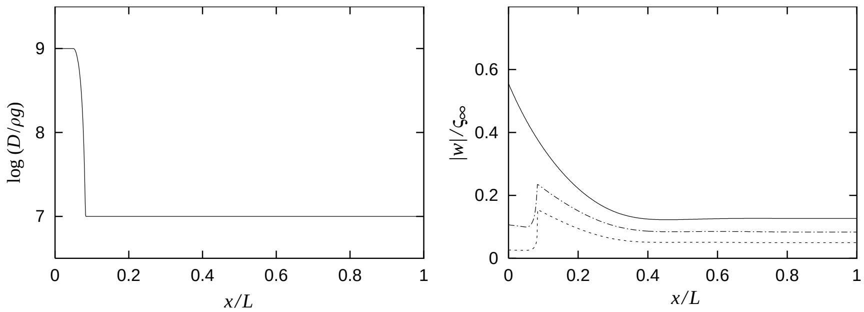

Fig. 6. Amplitude of deflection for $d = 10^7$ m^4 (top), and $d^{(1)} = 10^9$ m^4 (middle), 10^{11} m^4 (bottom) resp. with $d^{(2)} = 10^7$ m^4, $\lambda_0/L = 0.3$ (with $L = 300$ m) for a semi-infinite plate.

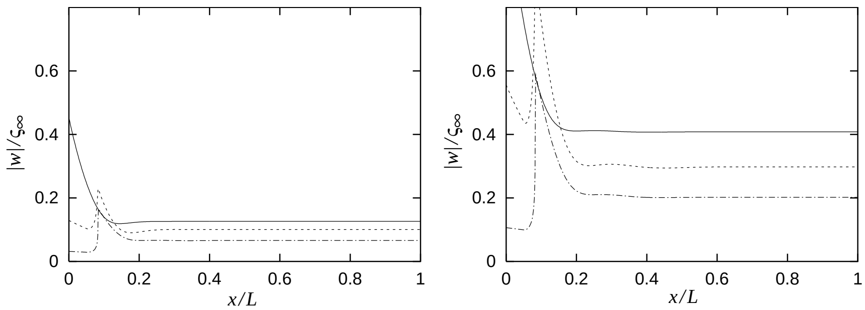

Fig. 7. Amplitude of deflection for $d = 10^5$ m^4 (top), and $d^{(1)} = 10^7$ m^4 (middle), 10^9 m^4 (bottom) resp. with $d^{(2)} = 10^5$ m^4, $\lambda_0/L = 0.1$ (left) and $\lambda_0/L = 0.3$ (right) for a semi-infinite plate ($L = 300$ m).

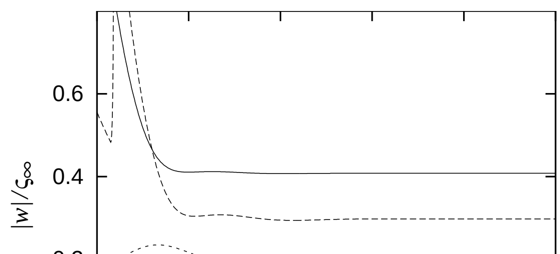

Fig. 8. From top to bottom: $c = 0$, $d = 10^5$ m^4; $c = 0$, $d^{(1)} = 10^7$ m^4 and $d^{(2)} = 10^5$ m^4; $c = 2$, $d = 10^5$ m^4; $c = 2$, $d^{(1)} = 10^7$ m^4 and $d^{(2)} = 10^5$ m^4.

values it varies as a half cosine wave

$$d = \{d^{(1)} + d^{(2)} + (d^{(1)} - d^{(2)})\cos((x - 15)\pi/10)\}/2,$$

see Fig. 6.

The results are a superposition of three modes, and one can clearly identify the travelling wave. The effect of the evanescent modes becomes negligible as soon as the amplitude of the deflection becomes constant. It can be seen also

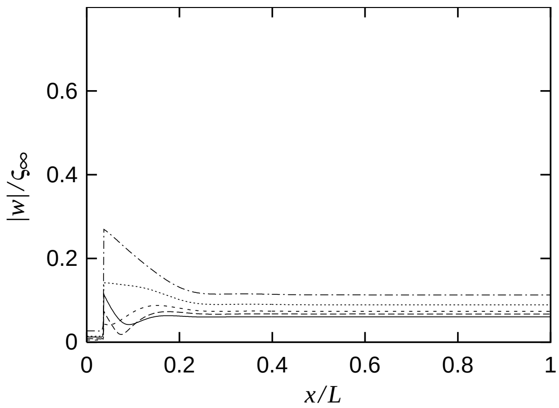

Fig. 9. From top to bottom: $c = 6$, $d^{(1)} = 10^{10}$ m^4, $d^{(2)} = 10^5$ m^4, $\lambda_0/L = 0.5, \ldots, 0.1$ m.

that the amplitude of the deflection becomes smaller if stiffness D of the left-hand end increases. This is in accordance with the findings of Takagi et al. (2000), who studied the effect of a rigid beam along the edge of the platform.

Fig. 7 illustrates results for a different value of the stiffness parameter D and two values of λ_0.

In Fig. 8, we show results for a case where the effects of a block with high rigidity and extra weight are taken into account. The extra weight is incorporated in the boundary condition at $x = 0$. The transition region from one value of the elastic parameter to the other is taken to be shorter than in Fig. 7. If we take a block with a length of 10 m and a height of 1 m the mass of one unit width of the material equals $c\rho$; in the computations we take $c = 2$ or 6, respectively.

One may conclude that the addition of extra weight in combination with high rigidity on the deflection is substantial. For each value of mass, one has to take an optimal value of the mass of the block. The figure shown concerns the transition of a very high to a very low rigidity. If we compare this result with the case of a constant low value one notices that the effect of the block is substantial. In Fig. 9, the influence of the wavelength is indicated.

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